

Operator representations on quantum spaces

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Abstract. In this article we present explicit formulae for q -differentiation on quantum spaces which could be of particular importance in physics, i.e., q -deformed Minkowski space and q -deformed Euclidean space in three or four dimensions. The calculations are based on the covariant differential calculus of these quantum spaces. Furthermore, our formulae can be regarded as a generalization of Jackson's q -derivative to three and four dimensions.

1 Introduction

One might say that the ideas of differential calculus are as old as physical science itself. Since its invention by Newton and Leibniz, an essential change of these ideas has not been necessary in the past. Although this can be seen as a great success, one cannot ignore the fact that up to now physicists have not been able to present a unified description of nature by using this traditional tool, i.e., a theory which does not break down at any possible space-time distances.

Quantum spaces, however, which are defined as co-module algebras of quantum groups and which can be interpreted as deformations of ordinary co-ordinate algebras [1] could provide a proper framework for developing a new kind of non-commutative analysis [2,3]. For our purposes it is sufficient to consider a quantum space as an algebra \mathcal{A}_q of formal power series in the non-commuting co-ordinates X_1, X_2, \dots, X_n ,

$$\mathcal{A}_q = \mathbb{C}[[X_1, \dots, X_n]]/\mathcal{I}, \quad (1)$$

where \mathcal{I} denotes the ideal generated by the relations of the non-commuting co-ordinates.

The algebra \mathcal{A}_q satisfies the Poincaré–Birkhoff–Witt property, i.e., the dimension of the subspace of homogeneous polynomials should be the same as for commuting co-ordinates. This property is the deeper reason why the monomials of normal ordering $X_1 X_2 \dots X_n$ constitute a basis of \mathcal{A}_q . In particular, we can establish a vector space isomorphism between \mathcal{A}_q and the commutative algebra \mathcal{A} generated by ordinary co-ordinates x_1, x_2, \dots, x_n :

$$\begin{aligned} \mathcal{W} : \mathcal{A} &\longrightarrow \mathcal{A}_q, \\ \mathcal{W}(x_1^{i_1} \dots x_n^{i_n}) &= X_1^{i_1} \dots X_n^{i_n}. \end{aligned} \quad (2)$$

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This vector space isomorphism can be extended to an algebra isomorphism introducing a non-commutative product in \mathcal{A} , the so-called \star -product [4,5]. This product is defined by the relation

$$\mathcal{W}(f \star g) = \mathcal{W}(f) \cdot \mathcal{W}(g), \quad (3)$$

where f and g are formal power series in \mathcal{A} . In [6] we have calculated the \star -product for quantum spaces which could be of particular importance in physics, i.e., q -deformed Minkowski space and q -deformed Euclidean space in three or four dimensions.

Additionally, for each of these quantum spaces exists a symmetry algebra [7,8] and a covariant differential calculus [9], which can provide an action upon the quantum spaces under consideration. By means of the relation

$$\mathcal{W}(h \triangleright f) := h \triangleright \mathcal{W}(f), \quad h \in \mathcal{H}, f \in \mathcal{A}, \quad (4)$$

we are also able to introduce an action upon the corresponding commutative algebra.

It is now our aim to present explicit formulae for the action of the partial derivatives on these spaces. In addition we have worked out representations of the generators of the q -deformed Lorentz algebra and the algebra of q -deformed angular momentum in three or four dimensions. All explicit formulae belong to left representations, as every right representation can be deduced from a left one by applying some simple rules.

2 q -Deformed Euclidean space in three dimensions

The q -deformed Euclidean space in three dimensions is spanned by the non-commuting co-ordinates X^+, X^3, X^- . Their commutation relations with the partial derivatives $\partial^+, \partial^3, \partial^-$ can be written in the general form [10]

$$\partial^A X^B = g^{AB} + (\hat{R}^{-1})_{CD}^{AB} X^C \partial^D, \quad (5)$$

$$A, B, C, D = 3, \pm,$$

where \hat{R}^{-1} denotes the inverse of the R -matrix of the quantum group $SO_q(3)$ and g^{AB} is the corresponding metric. Explicitly we have

$$\partial^+ X^+ = X^+ \partial^+, \quad (6)$$

$$\partial^+ X^3 = q^2 X^3 \partial^+ - q^2 \lambda \lambda_+ X^+ \partial^3,$$

$$\partial^+ X^- = -q + q^4 X^- \partial^+ - q^3 \lambda \lambda_+ X^3 \partial^3 + q^3 \lambda^2 \lambda_+ X^+ \partial^-,$$

$$\partial^3 X^+ = q^2 X^+ \partial^3, \quad (7)$$

$$\partial^3 X^3 = 1 + q^2 X^3 \partial^3 - q^3 \lambda \lambda_+ X^+ \partial^-,$$

$$\partial^3 X^- = q^2 X^- \partial^3 - q^2 \lambda \lambda_+ X^3 \partial^-,$$

$$\partial^- X^+ = -q^{-1} + q^4 X^+ \partial^-, \quad (8)$$

$$\partial^- X^3 = q^2 X^3 \partial^-,$$

$$\partial^- X^- = X^- \partial^-,$$

with $\lambda = q - q^{-1}$ and $\lambda_+ = q + q^{-1}$. On the q -deformed version of 3-dimensional Euclidean space there exists a second covariant differential calculus. Its defining relations read

$$\hat{\partial}^A X^B = g^{AB} + (\hat{R})_{CD}^{AB} X^C \hat{\partial}^D, \quad A, B, C, D = 3, \pm. \quad (9)$$

Written out, we get

$$\hat{\partial}^+ X^+ = X^+ \hat{\partial}^+, \quad (10)$$

$$\hat{\partial}^+ X^3 = q^{-2} X^3 \hat{\partial}^+,$$

$$\hat{\partial}^+ X^- = -q + q^{-4} X^- \hat{\partial}^+,$$

$$\hat{\partial}^3 X^+ = q^{-2} X^+ \hat{\partial}^3 + q^{-2} \lambda \lambda_+ X^3 \hat{\partial}^+, \quad (11)$$

$$\hat{\partial}^3 X^3 = 1 + q^{-2} X^3 \hat{\partial}^3 + q^{-3} \lambda \lambda_+ X^- \hat{\partial}^+,$$

$$\hat{\partial}^3 X^- = q^{-2} X^- \hat{\partial}^3,$$

$$\hat{\partial}^- X^+ = -q^{-1} + q^{-4} X^+ \hat{\partial}^- + q^{-3} \lambda \lambda_+ X^3 \hat{\partial}^3 + q^{-3} \lambda^{-2} \lambda_+ X^- \hat{\partial}^+, \quad (12)$$

$$\hat{\partial}^- X^3 = q^{-2} X^3 \hat{\partial}^- + q^{-2} \lambda \lambda_+ X^- \hat{\partial}^3,$$

$$\hat{\partial}^- X^- = X^- \hat{\partial}^-.$$

These relations yield a Hopf structure for the two sets of derivatives which can be derived by the same method already explained in [21]. Using the generators of angular momentum¹ L^+ , L^- , $\tau^{-1/2}$ and the scaling operator Λ [10] one obtains in the first case for the co-product Δ , the antipode S and the co-unit ε the following expressions:

$$\Delta(\partial^-) = \partial^- \otimes 1 + \Lambda^{\frac{1}{2}} \tau^{-\frac{1}{2}} \otimes \partial^-, \quad (13)$$

¹ One has to pay attention to the different normalisation of the L^A . Analogous to [11] we made the substitution $L^A \rightarrow -q^{-3} L^A$

$$\Delta(\partial^3) = \partial^3 \otimes 1 + \Lambda^{\frac{1}{2}} \otimes \partial^3 + \lambda \lambda_+ \Lambda^{\frac{1}{2}} L^+ \otimes \partial^-,$$

$$\Delta(\partial^+) = \partial^+ \otimes 1 + \Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} \otimes \partial^+ + q \lambda \lambda_+ \Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} L^+ \otimes \partial^3 + q^2 \lambda^2 \lambda_+ \Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} (L^+)^2 \otimes \partial^-,$$

$$S(\partial^-) = -\Lambda^{-\frac{1}{2}} \tau^{\frac{1}{2}} \partial^-, \quad (14)$$

$$S(\partial^3) = -\Lambda^{-\frac{1}{2}} \partial^3 + q^2 \lambda \lambda_+ \Lambda^{-\frac{1}{2}} \tau^{\frac{1}{2}} L^+ \partial^-,$$

$$S(\partial^+) = -\Lambda^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \partial^+ + q \lambda \lambda_+ \Lambda^{-\frac{1}{2}} L^+ \partial^3 - q^4 \lambda^2 \lambda_+ \Lambda^{-\frac{1}{2}} \tau^{\frac{1}{2}} (L^+)^2 \partial^-,$$

$$\varepsilon(\partial^+) = \varepsilon(\partial^3) = \varepsilon(\partial^-) = 0. \quad (15)$$

In the second case the Hopf structure is given by

$$\Delta(\hat{\partial}^+) = \hat{\partial}^+ \otimes 1 + \Lambda^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \otimes \hat{\partial}^+, \quad (16)$$

$$\Delta(\hat{\partial}^3) = \hat{\partial}^3 \otimes 1 + \Lambda^{-\frac{1}{2}} \otimes \hat{\partial}^3 + \lambda \lambda_+ \Lambda^{-\frac{1}{2}} L^- \otimes \hat{\partial}^+,$$

$$\Delta(\hat{\partial}^-) = \hat{\partial}^- \otimes 1 + \Lambda^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \otimes \hat{\partial}^- + q^{-1} \lambda \lambda_+ \Lambda^{-\frac{1}{2}} \tau^{\frac{1}{2}} L^- \otimes \hat{\partial}^3 + q^{-2} \lambda^2 \lambda_+ \Lambda^{-\frac{1}{2}} \tau^{\frac{1}{2}} (L^-)^2 \otimes \hat{\partial}^+,$$

$$S(\hat{\partial}^+) = -\Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} \hat{\partial}^+, \quad (17)$$

$$S(\hat{\partial}^3) = -\Lambda^{\frac{1}{2}} \hat{\partial}^3 + q^{-2} \lambda \lambda_+ \Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} L^- \hat{\partial}^+,$$

$$S(\hat{\partial}^-) = -\Lambda^{\frac{1}{2}} \tau^{-\frac{1}{2}} \hat{\partial}^- + q^{-1} \lambda \lambda_+ \Lambda^{\frac{1}{2}} L^- \hat{\partial}^3 - q^{-4} \lambda^2 \lambda_+ \Lambda^{\frac{1}{2}} \tau^{\frac{1}{2}} (L^-)^2 \hat{\partial}^+,$$

$$\varepsilon(\hat{\partial}^+) = \varepsilon(\hat{\partial}^3) = \varepsilon(\hat{\partial}^-) = 0. \quad (18)$$

Due to the relation

$$\partial^A \triangleright (f \star g) = (\partial_{(1)}^A \triangleright f) \star (\partial_{(2)}^A \triangleright g), \quad (19)$$

the Leibniz rules for products of arbitrary power series can be read off from the co-product $\Delta(\partial^A) = \partial_{(1)}^A \otimes \partial_{(2)}^A$ quite easily [12, 13].

For applying this formula, however, it is necessary to know the representations of the generators L^+ , L^- , $\tau^{-1/2}$ and the scaling operator Λ , which can be computed from the commutation relations

$$L^+ X^+ = X^+ L^+, \quad (20)$$

$$L^+ X^3 = X^3 L^+ - q X^+ \tau^{-\frac{1}{2}},$$

$$L^+ X^- = X^- L^+ - X^3 \tau^{-\frac{1}{2}},$$

$$L^- X^+ = X^+ L^- + X^3 \tau^{-\frac{1}{2}}, \quad (21)$$

$$L^- X^3 = X^3 L^- + q^{-1} X^- \tau^{-\frac{1}{2}},$$

$$L^- X^- = X^- L^-,$$

$$\tau^{-\frac{1}{2}} X^\pm = q^{\pm 2} X^\pm \tau^{-\frac{1}{2}}, \quad (22)$$

$$\begin{aligned} \tau^{-\frac{1}{2}} X^3 &= X^3 \tau^{-\frac{1}{2}}, & \cdot \left(D_{q^{-4}}^+ D_{q^{-4}}^- \right)^i f(\underline{x}). \\ \Lambda^{\frac{1}{2}} X^A &= q^2 X^A \Lambda^{\frac{1}{2}}, \quad A = \pm, 3. \end{aligned} \tag{28}$$

To calculate the explicit form of their action on the quantum space algebra we iterate the action of the generators on monomials of normal ordering $X^+ X^3 X^-$ until all generators have moved to the right. With the relation $T \triangleright 1 = \varepsilon(T)$ and after a possible normal ordering the wanted representations follow immediately. Such calculations can also be found in [14]. Finally, in the sense of definition (4) the action of the generators L^+ , L^- , $\tau^{-1/2}$ and the scaling operator Λ take the form²

$$\begin{aligned} L^+ \triangleright f &= -q^2 x^3 (D_{q^4}^- f)(q^{-2} x^-) - q x^+ (D_{q^2}^3 f)(q^{-2} x^-), \\ L^- \triangleright f &= x^3 (D_{q^4}^+ f)(q^{-2} x^-) + q^{-1} x^- (D_{q^2}^3 f)(q^{-2} x^-), \\ \tau^{\pm \frac{1}{2}} \triangleright f &= f(q^{\mp 2} x^+, q^{\pm 2} x^-), \\ \Lambda^{\pm \frac{1}{2}} \triangleright f &= f(q^{\pm 2} x^+, q^{\pm 2} x^3, q^{\pm 2} x^-). \end{aligned} \tag{23}$$

Similar expressions can be derived for the partial derivatives ∂^+ , ∂^3 , ∂^- with the end result

$$\begin{aligned} \partial^- \triangleright f &= -q^{-1} D_{q^4}^+ f, \\ \partial^3 \triangleright f &= D_{q^2}^3 f(q^2 x^+), \\ \partial^+ \triangleright f &= -q D_{q^4}^- f(q^2 x^3) - q \lambda x^+ (D_{q^2}^3)^2 f. \end{aligned} \tag{24}$$

In the case of the second differential calculus the representations of the partial derivatives take on a very simple form, if they refer to the ordering $X^- X^3 X^+$. For this new ordering we have

$$\begin{aligned} \hat{\partial}^+ \tilde{\triangleright} f &= -q D_{q^{-4}}^- f, \\ \hat{\partial}^3 \tilde{\triangleright} f &= D_{q^{-2}}^3 f(q^{-2} x^-), \\ \hat{\partial}^- \tilde{\triangleright} f &= -q^{-1} D_{q^{-4}}^+ f(q^{-2} x^3) + q^{-1} \lambda x^- (D_{q^{-2}}^3)^2 f. \end{aligned} \tag{25}$$

In addition, we give the identities

$$\begin{aligned} \partial^A \triangleright (\hat{U}^{-1} f) &= \hat{U}^{-1} (\partial^A \tilde{\triangleright} f), \\ \hat{\partial}^A \tilde{\triangleright} (\hat{U} f) &= \hat{U} (\hat{\partial}^A \triangleright f), \end{aligned} \tag{26}$$

where³

$$\begin{aligned} \hat{U}^{-1} f &= \sum_{i=0}^{\infty} \lambda^i \frac{(x^3)^{2i}}{[[i]]_{q^4}!} q^{2\hat{n}_3(\hat{n}_+ + \hat{n}_- + i)} \\ &\cdot \left(D_{q^4}^+ D_{q^4}^- \right)^i f(\underline{x}), \\ \hat{U} f &= \sum_{i=0}^{\infty} (-\lambda)^i \frac{(x^3)^{2i}}{[[i]]_{q^{-4}}!} q^{-2\hat{n}_3(\hat{n}_+ + \hat{n}_- + i)} \end{aligned} \tag{27}$$

With these formulae at hand, which can easily be derived from the considerations in [6], we are in a position to deal with representations of one given ordering, as the operators \hat{U}^{-1} and \hat{U} transform functions of ordering $X^- X^3 X^+$ to the corresponding ones of ordering $X^+ X^3 X^-$ and vice versa.

All representations considered so far have been computed by commuting the acting generators from the left side of a monomial to the right. These representations are thus called left representations. However, if we commute the acting generators from the right side of a monomial to the left, right representations will consequently arise. But these right representations can be read off from left ones quite easily, as right representations of a generator are always linked to left ones via the identity

$$\overline{\partial^A \triangleright f} = \bar{f} \triangleleft \overline{\partial^A}. \tag{29}$$

From the conjugation properties [10]

$$\overline{X^\pm} = -q^{\pm 1} X^\mp, \tag{30}$$

$$\overline{X^3} = X^3,$$

$$\overline{\partial^\mp} = -q^{-5} \hat{\partial}^-, \tag{31}$$

$$\overline{\partial^3} = -q^{-6} \hat{\partial}^3,$$

$$\overline{\partial^+} = -q^{-7} \hat{\partial}^+,$$

$$\overline{L^\pm} = -q^{\pm 1} L^\mp, \tag{32}$$

one obtains the translation rules

$$f \triangleleft L^\pm \overset{\leftrightarrow}{\longleftarrow} L^\mp \triangleright f, \tag{33}$$

$$f \triangleleft \partial^\pm \overset{\leftrightarrow}{\longleftarrow} -q^{-6} \hat{\partial}^\mp \triangleright f, \tag{34}$$

$$f \triangleleft \partial^3 \overset{\leftrightarrow}{\longleftarrow} -q^{-6} \hat{\partial}^3 \triangleright f,$$

$$f \triangleleft \hat{\partial}^\pm \overset{\leftrightarrow}{\longleftarrow} -q^6 \partial^\mp \triangleright f, \tag{35}$$

$$f \triangleleft \hat{\partial}^3 \overset{\leftrightarrow}{\longleftarrow} -q^6 \partial^3 \triangleright f,$$

where the symbol $\overset{\leftrightarrow}{\longleftarrow}$ denotes that one can make a transition between the two expressions by applying the substitutions

$$x^\pm \rightarrow x^\mp, \quad D_{q^a}^\pm \rightarrow D_{q^a}^\mp, \quad \hat{n}^\pm \rightarrow \hat{n}^\mp. \tag{36}$$

The following shall serve as an example:

$$x^+ x^- (D_q^+)^2 D_q^- f(q^2 x^-) \tag{37}$$

$$\overset{\leftrightarrow}{\longleftarrow} x^- x^+ (D_q^-)^2 D_q^+ f(q^2 x^+).$$

The right representations of the generators τ^3 and Λ are derived most easily from the identity

$$f \triangleleft h = S^{-1}(h) \triangleright f; \tag{38}$$

² For notation see Appendix A

³ For notation see again Appendix A

hence

$$f \triangleleft \tau^{\pm \frac{1}{2}} = S^{-1} \left(\tau^{\pm \frac{1}{2}} \right) \triangleright f = \tau^{\pm \frac{1}{2}} \triangleright f, \quad (39)$$

$$f \triangleleft \Lambda^{\pm \frac{1}{2}} = S^{-1} \left(\Lambda^{\pm \frac{1}{2}} \right) \triangleright f = \Lambda^{\mp \frac{1}{2}} \triangleright f.$$

Finally, let us remark that due to the relation

$$f \partial^A = \partial_{(2)}^A (f \triangleleft \partial_{(1)}^A) \quad (40)$$

again the co-products of the two differential calculi directly yield Leibniz rules for right representations of the partial derivatives.

3 q -Deformed Euclidean space in four dimensions

The 4-dimensional q -deformed Euclidean space can be treated in very much the same way as the 3-dimensional case. For the relations between partial derivatives and coordinates we now have

$$\begin{aligned} \partial^i X^j &= g^{ij} + q(\hat{R}^{-1})_{kl}^{ij} X^k \partial^l, \\ i, j, k, l &= 1, \dots, 4, \end{aligned} \quad (41)$$

where g^{ij} denotes the 4-dimensional Euclidean quantum space metric and \hat{R} the R -matrix of $SO_q(4)$. With the notation in [12] these relations read explicitly

$$\partial^1 X^1 = X^1 \partial^1, \quad (42)$$

$$\partial^1 X^2 = qX^2 \partial^1,$$

$$\partial^1 X^3 = qX^3 \partial^1,$$

$$\partial^1 X^4 = q^{-1} + q^2 X^4 \partial^1,$$

$$\partial^2 X^1 = qX^1 \partial^2 - q\lambda X^2 \partial^1, \quad (43)$$

$$\partial^2 X^2 = X^2 \partial^2,$$

$$\partial^2 X^3 = 1 + q^2 X^3 \partial^2 + q^2 \lambda X^4 \partial^1,$$

$$\partial^2 X^4 = qX^4 \partial^2,$$

$$\partial^3 X^1 = qX^1 \partial^3 - q\lambda X^3 \partial^1, \quad (44)$$

$$\partial^3 X^2 = 1 + q^2 X^2 \partial^3 + q^2 \lambda X^4 \partial^1,$$

$$\partial^3 X^3 = X^3 \partial^3,$$

$$\partial^3 X^4 = qX^4 \partial^3,$$

$$\partial^4 X^1 = q + q^2 X^1 \partial^4 + q^2 \lambda (X^2 \partial^3 + X^3 \partial^2 + \lambda X^4 \partial^1),$$

$$\partial^4 X^2 = qX^2 \partial^4 - q\lambda X^4 \partial^2, \quad (45)$$

$$\partial^4 X^3 = qX^3 \partial^4 - q\lambda X^4 \partial^3,$$

$$\partial^4 X^4 = X^4 \partial^4.$$

For the second set of derivatives the following relations hold:

$$\partial^i X^j = g^{ij} + q^{-1}(\hat{R})_{kl}^{ij} X^k \partial^l, \quad (46)$$

$$i, j, k, l = 1, \dots, 4.$$

In a more explicit form one can write

$$\hat{\partial}^1 X^1 = X^1 \hat{\partial}^1, \quad (47)$$

$$\hat{\partial}^1 X^2 = q^{-1} X^2 \hat{\partial}^1 + q^{-1} \lambda X^1 \hat{\partial}^2,$$

$$\hat{\partial}^1 X^3 = q^{-1} X^3 \hat{\partial}^1 + q^{-1} \lambda X^1 \hat{\partial}^3,$$

$$\hat{\partial}^1 X^4 = q^{-1} + q^{-2} X^4 \hat{\partial}^1 - q^{-2} \lambda (X^2 \hat{\partial}^3 + X^3 \hat{\partial}^2 - \lambda X^1 \hat{\partial}^4),$$

$$\hat{\partial}^2 X^1 = q^{-1} X^1 \hat{\partial}^2, \quad (48)$$

$$\hat{\partial}^2 X^2 = X^2 \hat{\partial}^2,$$

$$\hat{\partial}^2 X^3 = 1 + q^{-2} X^3 \hat{\partial}^2 - q^{-2} \lambda X^1 \hat{\partial}^4,$$

$$\hat{\partial}^2 X^4 = q^{-1} X^4 \hat{\partial}^2 + q^{-1} \lambda X^2 \hat{\partial}^4,$$

$$\hat{\partial}^3 X^1 = q^{-1} X^1 \hat{\partial}^3, \quad (49)$$

$$\hat{\partial}^3 X^2 = 1 + q^{-2} X^2 \hat{\partial}^3 - q^{-2} \lambda X^1 \hat{\partial}^4,$$

$$\hat{\partial}^3 X^3 = X^3 \hat{\partial}^3,$$

$$\hat{\partial}^3 X^4 = q^{-1} X^4 \hat{\partial}^3 + q^{-1} \lambda X^3 \hat{\partial}^4,$$

$$\hat{\partial}^4 X^1 = q + q^{-2} X^1 \hat{\partial}^4, \quad (50)$$

$$\hat{\partial}^4 X^2 = q^{-1} X^2 \hat{\partial}^4,$$

$$\hat{\partial}^4 X^3 = q^{-1} X^3 \hat{\partial}^4,$$

$$\hat{\partial}^4 X^4 = X^4 \hat{\partial}^4.$$

From these relations we again can deduce a Hopf structure for the derivatives ∂^i , $i = 1, \dots, 4$, which in terms of the $U_q(so_4)$ generators L_i^{\pm} , K_i ($i = 1, 2$) and the scaling operator Λ becomes

$$\Delta(\partial^1) = \partial^1 \otimes 1 + \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \partial^1, \quad (51)$$

$$\Delta(\partial^2) = \partial^2 \otimes 1 + \Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \partial^2$$

$$+ q\lambda \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^+ \otimes \partial^1,$$

$$\Delta(\partial^3) = \partial^3 \otimes 1 + \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{-\frac{1}{2}} \otimes \partial^3$$

$$+ q\lambda \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_2^+ \otimes \partial^1,$$

$$\Delta(\partial^4) = \partial^4 \otimes 1 + \Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{-\frac{1}{2}} \otimes \partial^4$$

$$- q^2 \lambda^2 \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^+ L_2^+ \otimes \partial^1$$

$$- q\lambda \Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} L_2^+ \otimes \partial^2$$

$$- q\lambda \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{-\frac{1}{2}} L_1^+ \otimes \partial^3,$$

$$S(\partial^1) = -\Lambda^{-\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{-\frac{1}{2}} \partial^1, \quad (52)$$

$$S(\partial^2) = -\Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{-\frac{1}{2}} (\partial^2 - q^2 \lambda L_1^+ \partial^1),$$

$$\begin{aligned}
 S(\partial^3) &= -\Lambda^{-\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} (\partial^3 - q^2 \lambda L_2^+ \partial^1), \\
 S(\partial^4) &= -\Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \partial^4 \\
 &\quad - q^2 \lambda \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} (L_1^+ \partial^3 + L_2^+ \partial^2) \\
 &\quad - q^4 \lambda^2 \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^+ L_2^+ \partial^1, \\
 \varepsilon(\partial^1) &= \varepsilon(\partial^2) = \varepsilon(\partial^3) = \varepsilon(\partial^4) = 0.
 \end{aligned} \tag{53}$$

In the same manner we get for the other derivatives $\hat{\partial}^i$, $i = 1, \dots, 4$,

$$\begin{aligned}
 \Delta(\hat{\partial}^1) &= \hat{\partial}^1 \otimes 1 + \Lambda^{-\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{-\frac{1}{2}} \otimes \hat{\partial}^1 \\
 &\quad - q^{-2} \lambda^2 \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^- L_2^- \otimes \hat{\partial}^4 \\
 &\quad - q^{-1} \lambda \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{-\frac{1}{2}} L_1^- \otimes \hat{\partial}^2 \\
 &\quad - q^{-1} \lambda \Lambda^{-\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} L_2^- \otimes \hat{\partial}^3, \\
 \Delta(\hat{\partial}^2) &= \hat{\partial}^2 \otimes 1 + \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{-\frac{1}{2}} \otimes \hat{\partial}^2 \\
 &\quad + q^{-1} \lambda \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_2^- \otimes \hat{\partial}^4, \\
 \Delta(\hat{\partial}^3) &= \hat{\partial}^3 \otimes 1 + \Lambda^{-\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \hat{\partial}^3 \\
 &\quad + q^{-1} \lambda \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^- \otimes \hat{\partial}^4, \\
 \Delta(\hat{\partial}^4) &= \hat{\partial}^4 \otimes 1 + \Lambda^{-\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \otimes \hat{\partial}^4, \\
 S(\hat{\partial}^1) &= -\Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} \hat{\partial}^1 \\
 &\quad - q^{-2} \lambda \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} (L_1^- \hat{\partial}^2 + L_2^- \hat{\partial}^3) \\
 &\quad + q^{-4} \lambda^2 \Lambda^{\frac{1}{2}} K_1^{\frac{1}{2}} K_2^{\frac{1}{2}} L_1^- L_2^- \hat{\partial}^4, \\
 S(\hat{\partial}^2) &= -\Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{\frac{1}{2}} (\hat{\partial}^2 - q^{-2} \lambda L_2^- \hat{\partial}^4), \\
 S(\hat{\partial}^3) &= -\Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{-\frac{1}{2}} (\hat{\partial}^3 - q^{-2} \lambda L_1^- \hat{\partial}^4), \\
 S(\hat{\partial}^4) &= -\Lambda^{\frac{1}{2}} K_1^{-\frac{1}{2}} K_2^{-\frac{1}{2}} \hat{\partial}^4, \\
 \varepsilon(\hat{\partial}^1) &= \varepsilon(\hat{\partial}^2) = \varepsilon(\hat{\partial}^3) = \varepsilon(\hat{\partial}^4) = 0.
 \end{aligned} \tag{55}$$

Note that the above expressions again yield Leibniz rules for products of functions, if the representations of the given $U_q(so_4)$ generators and the Λ operator are known. Toward this end we need their commutation relations with the quantum space coordinates, for which we have [15]

$$\begin{aligned}
 L_1^+ X^1 &= q X^1 L_1^+ - q^{-1} X^2, \\
 L_1^+ X^2 &= q^{-1} X^2 L_1^+, \\
 L_1^+ X^3 &= q X^3 L_1^+ + q^{-1} X^4, \\
 L_1^+ X^4 &= q^{-1} X^4 L_1^+, \\
 L_2^+ X^1 &= q X^1 L_2^+ - q^{-1} X^3,
 \end{aligned} \tag{57}$$

$$\begin{aligned}
 L_2^+ X^1 &= q X^1 L_2^+ - q^{-1} X^3,
 \end{aligned} \tag{58}$$

$$\begin{aligned}
 L_2^+ X^2 &= q X^2 L_2^+ + q^{-1} X^4, \\
 L_2^+ X^3 &= q^{-1} X^3 L_2^+, \\
 L_2^+ X^4 &= q^{-1} X^4 L_2^+, \\
 L_1^- X^1 &= q X^1 L_1^-, \\
 L_1^- X^2 &= q^{-1} X^2 L_1^- - q X^1, \\
 L_1^- X^3 &= q X^3 L_1^-, \\
 L_1^- X^4 &= q^{-1} X^4 L_1^- + q X^3,
 \end{aligned} \tag{59}$$

$$\begin{aligned}
 L_2^- X^1 &= q X^1 L_2^-, \\
 L_2^- X^2 &= q X^2 L_2^-, \\
 L_2^- X^3 &= q^{-1} X^3 L_2^- - q X^1, \\
 L_2^- X^4 &= q^{-1} X^4 L_2^- + q X^2,
 \end{aligned} \tag{60}$$

$$\begin{aligned}
 K_1 X^1 &= q^{-1} X^1 K_1, \\
 K_1 X^2 &= q X^2 K_1, \\
 K_1 X^3 &= q^{-1} X^3 K_1, \\
 K_1 X^4 &= q X^4 K_1,
 \end{aligned} \tag{61}$$

$$\begin{aligned}
 K_2 X^1 &= q^{-1} X^1 K_2, \\
 K_2 X^2 &= q^{-1} X^2 K_2, \\
 K_2 X^3 &= q X^3 K_2, \\
 K_2 X^4 &= q X^4 K_2,
 \end{aligned} \tag{62}$$

$$\Lambda X^i = q^2 X^i \Lambda, \quad i = 1, \dots, 4. \tag{63}$$

In normal ordering $X^1 X^2 X^3 X^4$ these relations lead to left representations of the following form:

$$\begin{aligned}
 L_1^+ \triangleright f &= x^4 D_{q^2}^3 f(qx^1, q^{-1}x^2, q^{-1}x^3) \\
 &\quad - x^2 D_{q^2}^1 f(q^{-1}x^1), \\
 L_2^+ \triangleright f &= x^4 D_{q^2}^2 f(qx^1, q^{-1}x^2, q^{-1}x^3) \\
 &\quad - x^3 D_{q^2}^1 f(q^{-1}x^1), \\
 L_1^- \triangleright f &= qx^3 D_{q^{-2}}^3 f(qx^1, q^{-1}x^2, qx^3) \\
 &\quad - qx^1 D_{q^{-2}}^2 f(qx^1), \\
 L_2^- \triangleright f &= qx^2 D_{q^{-2}}^4 f(qx^1, qx^2, q^{-1}x^3) \\
 &\quad - qx^1 D_{q^{-2}}^3 f(qx^1), \\
 K_1 \triangleright f &= f(q^{-1}x^1, qx^2, q^{-1}x^3, qx^4), \\
 K_2 \triangleright f &= f(q^{-1}x^1, q^{-1}x^2, qx^3, qx^4), \\
 \Lambda^{\pm \frac{1}{2}} \triangleright f &= f(q^{\pm 1}x^1, q^{\pm 1}x^2, q^{\pm 1}x^3, q^{\pm 1}x^4).
 \end{aligned} \tag{64}$$

Accordingly, the representations of the partial derivatives $\hat{\partial}^i$, $i = 1, \dots, 4$, can be written as

$$\begin{aligned} \hat{\partial}^1 \triangleright f &= q^{-1} D_{q^{-2}}^4 f(q^{-1} x^2, q^{-1} x^3) \\ &\quad + q^{-1} \lambda x^1 D_{q^{-2}}^2 D_{q^{-2}}^3 f, \\ \hat{\partial}^2 \triangleright f &= D_{q^{-2}}^3 f(q^{-1} x^1), \\ \hat{\partial}^3 \triangleright f &= D_{q^{-2}}^2 f(q^{-1} x^1), \\ \hat{\partial}^4 \triangleright f &= q D_{q^{-2}}^1 f. \end{aligned} \tag{65}$$

For the sake of simplicity the representations of the unhat- ted partial derivatives refer to a different ordering, namely $X^4 X^3 X^2 X^1$. In this setting they are given by

$$\begin{aligned} \hat{\partial}^i \tilde{\triangleright} f &= q D_{q^2}^1 f(qx^2, qx^3) - q \lambda x^4 D_{q^2}^2 D_{q^2}^3 f, \\ \hat{\partial}^3 \tilde{\triangleright} f &= D_{q^2}^2 f(qx^4), \\ \hat{\partial}^2 \tilde{\triangleright} f &= D_{q^2}^3 f(qx^4), \\ \hat{\partial}^1 \tilde{\triangleright} f &= q^{-1} D_{q^2}^4 f, \end{aligned} \tag{66}$$

and if we want to have representations belonging to one given ordering only, we can apply the formulae

$$\begin{aligned} \hat{\partial}^i \triangleright (\hat{U}^{-1} f) &= \hat{U}^{-1} (\hat{\partial}^i \tilde{\triangleright} f), \\ \hat{\partial}^i \tilde{\triangleright} (\hat{U} f) &= \hat{U} (\hat{\partial}^i \triangleright f). \end{aligned} \tag{67}$$

with

$$\begin{aligned} \hat{U}^{-1} f &= \sum_{i=0}^{\infty} \lambda^i \frac{(x^2 x^3)^i}{[[i]]_{q^{-2}}!} q^{-(\hat{n}_2 + \hat{n}_3)(\hat{n}_1 + \hat{n}_4 + i)} \\ &\quad \cdot \left(D_{q^{-2}}^4 D_{q^{-2}}^1 \right)^i f(\underline{x}), \\ \hat{U} f &= \sum_{i=0}^{\infty} (-\lambda)^i \frac{(x^2 x^3)^i}{[[i]]_{q^2}!} q^{(\hat{n}_2 + \hat{n}_3)(\hat{n}_1 + \hat{n}_4 + i)} \\ &\quad \cdot \left(D_{q^2}^1 D_{q^2}^4 \right)^i f(\underline{x}). \end{aligned} \tag{68}$$

Using the conjugation properties [15]

$$\begin{aligned} \overline{X^1} &= q^{-1} X^4, \\ \overline{X^2} &= X^3, \\ \overline{X^3} &= X^2, \\ \overline{X^4} &= q X^1, \\ \overline{\partial^1} &= -q^{-5} \hat{\partial}^4, \\ \overline{\partial^2} &= -q^{-4} \hat{\partial}^3, \\ \overline{\partial^3} &= -q^{-4} \hat{\partial}^2, \\ \overline{\partial^4} &= -q^{-3} \hat{\partial}^1, \end{aligned} \tag{70}$$

(71)

$$\overline{L_i^{\pm}} = q^{\mp 2} L_i^{\mp}, \quad i = 1, 2, \tag{72}$$

and taking the considerations mentioned in the last section into account we again find the translation rules

$$\begin{aligned} f \triangleleft \partial^i \xleftrightarrow{j \leftrightarrow j'} -q^{-4} \hat{\partial}^{i'} \triangleright f, \\ f \triangleleft \hat{\partial}^i \xleftrightarrow{j \leftrightarrow j'} -q^4 \partial^{i'} \triangleright f, \quad i = 1, \dots, 4, \\ f \triangleleft L_i^+ \xleftrightarrow{j \leftrightarrow j'} q^{-3} L_i^- \triangleright f, \\ f \triangleleft L_i^- \xleftrightarrow{j \leftrightarrow j'} q^3 L_i^+ \triangleright f, \quad i = 1, 2, \end{aligned} \tag{73}$$

where $i' = 5 - i$. The symbol $\xleftrightarrow{j \leftrightarrow j'}$ now indicates that one can make a transition between the two expressions by the substitution

$$x^j \longleftrightarrow x^{j'}, \quad D_{q^a}^j \longleftrightarrow D_{q^a}^{j'}, \tag{74}$$

where $j = 1, \dots, 4$, $j' = 5 - i$. An example shall illustrate this:

$$\begin{aligned} D_{q^2}^1 D_{q^2}^2 f(qx^1, q^2 x^3) \\ \xleftrightarrow{j \leftrightarrow j'} D_{q^2}^4 D_{q^2}^3 f(qx^4, q^2 x^2). \end{aligned} \tag{75}$$

Last but not least we have to treat the representations of the diagonal generators K_1, K_2, Λ , which can be derived from the identity (38) quite easily. Thus we have

$$f \triangleleft K_1 = (K_1)^{-1} \triangleright f, \tag{76}$$

$$f \triangleleft K_2 = (K_2)^{-1} \triangleright f,$$

$$f \triangleleft \Lambda^{\pm \frac{1}{2}} = \Lambda^{\mp \frac{1}{2}} \triangleright f. \tag{77}$$

4 q -Deformed Minkowski space

From a physical point of view q -deformed Minkowski space [16–18]⁴ is the most interesting one of all considered cases. In addition, a treatment is desirable which pays attention to the central time element X^0 [20]. The general form of the commutation relations between partial derivatives and space-time coordinates now reads [10]

$$\begin{aligned} \partial^A X^B &= \eta^{AB} + q^{-2} (\hat{R}_{\text{II}}^{-1})_{CD}^{AB} X^C \partial^D, \\ A, B, C, D &= 0, 3, \pm, \end{aligned} \tag{78}$$

where η^{AB} denotes the metric and \hat{R}_{II} one of the two R -matrices of q -deformed Minkowski spaces. For the sake of simplicity we introduce the light cone coordinate $\tilde{X}^3 = X^3 - X^0$ and the corresponding partial derivative $\tilde{\partial}^3 = \partial^3 - \partial^0$. In terms of these quantities the above relations become

$$\tilde{\partial}^3 \tilde{X}^3 = \tilde{X}^3 \tilde{\partial}^3, \tag{79}$$

⁴ For a different version of q -deformed Minkowski space see also [19]

$$\begin{aligned}
\tilde{\partial}^3 X^+ &= X^+ \tilde{\partial}^3 + q^{-1} \lambda \tilde{X}^3 \partial^+, \\
\tilde{\partial}^3 X^3 &= 1 + q^{-2} X^3 \tilde{\partial}^3 + q^{-2} \lambda \lambda_+^{-1} \tilde{X}^3 \tilde{\partial}^3 + q^{-2} \lambda X^- \partial^+, \\
\tilde{\partial}^3 X^- &= q^{-2} X^- \tilde{\partial}^3, \\
\partial^+ \tilde{X}^3 &= q^{-2} \tilde{X}^3 \partial^+, \\
\partial^+ X^+ &= X^+ \partial^+, \\
\partial^+ X^3 &= X^3 \partial^+ - \lambda \lambda_+^{-1} \tilde{X}^3 \partial^+ - \lambda \lambda_+^{-1} X^+ \tilde{\partial}^3, \\
\partial^+ X^- &= -q + q^{-2} X^- \partial^+ - q^{-1} \lambda \lambda_+^{-1} \tilde{X}^3 \partial^3,
\end{aligned} \tag{80}$$

$$\begin{aligned}
\partial^- \tilde{X}^3 &= \tilde{X}^3 \partial^- + q^{-1} \lambda X^- \tilde{\partial}^3, \\
\partial^- X^+ &= -q^{-1} + q^{-2} X^+ \partial^- + q^{-2} \lambda \tilde{X}^3 \partial^0 \\
&\quad + q^{-2} \lambda^2 X^- \partial^+ + q^{-2} \lambda X^3 \tilde{\partial}^3 \\
&\quad + q^{-1} \lambda \lambda_+^{-1} \tilde{X}^3 \tilde{\partial}^3, \\
\partial^- X^3 &= q^{-2} X^3 \partial^- + q^{-2} \lambda \lambda_+^{-1} \tilde{X}^3 \partial^- \\
&\quad + \lambda \lambda_+^{-1} (1 + 2q^{-2}) X^- \tilde{\partial}^3 + q^{-1} \lambda X^- \partial^0, \\
\partial^- X^- &= X^- \partial^-,
\end{aligned} \tag{81}$$

$$\begin{aligned}
\partial^0 \tilde{X}^3 &= 1 + q^{-2} \tilde{X}^3 \partial^0 + q^{-2} \lambda X^- \partial^+ - \lambda \lambda_+^{-1} \tilde{X}^3 \tilde{\partial}^3, \\
\partial^0 X^+ &= q^{-2} X^+ \partial^0 + q^{-1} \lambda X^3 \partial^+ \\
&\quad - \lambda \lambda_+^{-1} \tilde{X}^3 \partial^+ - \lambda \lambda_+^{-1} X^+ \tilde{\partial}^3, \\
\partial^0 X^3 &= X^3 \partial^0 - \lambda \lambda_+^{-1} \tilde{X}^3 \partial^0 \\
&\quad + q^{-1} \lambda \lambda_+^{-1} (X^- \partial^+ - X^+ \partial^-) \\
&\quad + q^{-2} \lambda \lambda_+^{-1} X^3 \tilde{\partial}^3 - \lambda \lambda_+^{-1} \tilde{X}^3 \tilde{\partial}^3, \\
\partial^0 X^- &= X^- \partial^0 + q^{-2} \lambda \lambda_+^{-1} X^- \tilde{\partial}^3 - \lambda \lambda_+^{-1} \tilde{X}^3 \partial^-.
\end{aligned} \tag{82}$$

For the second differential calculus we have the relations

$$\hat{\partial}^A X^B = \eta^{AB} + q^2 (\hat{R}_{\text{II}})_{CD}^{AB} X^C \hat{\partial}^D, \tag{83}$$

which gives in a more explicit form

$$\begin{aligned}
\hat{\partial}^3 \tilde{X}^3 &= \tilde{X}^3 \hat{\partial}^3, \\
\hat{\partial}^3 X^- &= X^- \hat{\partial}^3 - q \lambda \tilde{X}^3 \hat{\partial}^-, \\
\hat{\partial}^3 X^3 &= 1 + q^2 X^3 \hat{\partial}^3 - q^2 \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^3 - q^2 \lambda X^+ \hat{\partial}^-, \\
\hat{\partial}^3 X^+ &= q^2 X^+ \hat{\partial}^3, \\
\hat{\partial}^- \tilde{X}^3 &= q^2 \tilde{X}^3 \hat{\partial}^-, \\
\hat{\partial}^- X^- &= X^- \hat{\partial}^-, \\
\hat{\partial}^- X^3 &= X^3 \hat{\partial}^- + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^- + \lambda \lambda_+^{-1} X^- \hat{\partial}^3, \\
\hat{\partial}^- X^+ &= -q^{-1} + q^2 X^+ \hat{\partial}^- + q \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^3,
\end{aligned} \tag{85}$$

$$\begin{aligned}
\hat{\partial}^+ \tilde{X}^3 &= \tilde{X}^3 \hat{\partial}^+ - q \lambda X^+ \hat{\partial}^3, \\
\hat{\partial}^+ X^- &= -q + q^2 X^- \hat{\partial}^+ - q^2 \lambda \tilde{X}^3 \hat{\partial}^0 + q^2 \lambda^2 X^+ \hat{\partial}^- \\
&\quad - q^2 \lambda X^3 \hat{\partial}^3 - q \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^3, \\
\hat{\partial}^+ X^3 &= q^2 X^3 \hat{\partial}^+ - q^2 \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^+ \\
&\quad - \lambda \lambda_+^{-1} (1 + 2q^2) X^+ \hat{\partial}^3 - q \lambda X^+ \hat{\partial}^0, \\
\hat{\partial}^+ X^+ &= X^+ \hat{\partial}^+,
\end{aligned} \tag{86}$$

$$\begin{aligned}
\hat{\partial}^0 \tilde{X}^3 &= 1 + q^2 \tilde{X}^3 \hat{\partial}^0 - q^2 \lambda X^+ \hat{\partial}^- + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^3, \\
\hat{\partial}^0 X^- &= q^2 X^- \hat{\partial}^0 - q \lambda X^3 \hat{\partial}^- \\
&\quad + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^- + \lambda \lambda_+^{-1} X^- \hat{\partial}^3, \\
\hat{\partial}^0 X^3 &= X^3 \hat{\partial}^0 + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^0 + q \lambda \lambda_+^{-1} (X^- \hat{\partial}^+ - X^+ \hat{\partial}^-) \\
&\quad - q^2 \lambda \lambda_+^{-1} X^3 \hat{\partial}^3 + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^3, \\
\hat{\partial}^0 X^+ &= X^+ \hat{\partial}^0 - q^2 \lambda \lambda_+^{-1} X^+ \hat{\partial}^3 + \lambda \lambda_+^{-1} \tilde{X}^3 \hat{\partial}^+.
\end{aligned} \tag{87}$$

From these relations we can again deduce a Hopf structure [21]. In terms of Lorentz generators T^+ , T^- , τ^3 , T^2 , S^1 , τ^1 , σ^2 and the scaling operator Λ the Hopf structure of the derivatives ∂^0 , ∂^+ , ∂^- , $\tilde{\partial}^3$ becomes

$$\begin{aligned}
\Delta(\tilde{\partial}^3) &= \tilde{\partial}^3 \otimes 1 + \Lambda^{\frac{1}{2}} \tau^1 \otimes \tilde{\partial}^3 \\
&\quad - q^{\frac{1}{2}} \lambda_+^{\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} (\tau^3)^{-\frac{1}{2}} S^1 \otimes \partial^+, \\
\Delta(\partial^+) &= \partial^+ \otimes 1 + \Lambda^{\frac{1}{2}} (\tau^3)^{-\frac{1}{2}} \sigma^2 \otimes \partial^+ \\
&\quad - q^{\frac{3}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} T^2 \otimes \tilde{\partial}^3, \\
\Delta(\partial^-) &= \partial^- \otimes 1 + \Lambda^{\frac{1}{2}} (\tau^3)^{\frac{1}{2}} \tau^1 \otimes \partial^- \\
&\quad - q^{-\frac{1}{2}} \lambda_+^{\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} S^1 \otimes \partial^0 \\
&\quad - \lambda^2 \Lambda^{\frac{1}{2}} (\tau^3)^{-\frac{1}{2}} T^- S^1 \otimes \partial^+ \\
&\quad + q^{-\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} \tau^1 T^- \otimes \tilde{\partial}^3 \\
&\quad - q^{-\frac{3}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} S^1 \otimes \tilde{\partial}^3, \\
\Delta(\partial^0) &= \partial^0 \otimes 1 + \Lambda^{\frac{1}{2}} \sigma^2 \otimes \partial^0 \\
&\quad - q^{\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} T^2 (\tau^3)^{\frac{1}{2}} \otimes \partial^- \\
&\quad + q^{\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} (\tau^3)^{-\frac{1}{2}} T^- \sigma^2 \otimes \partial^+ \\
&\quad + q^{\frac{3}{2}} \lambda_+^{-\frac{1}{2}} \lambda \Lambda^{\frac{1}{2}} (\tau^3)^{-\frac{1}{2}} S^1 \otimes \partial^+ \\
&\quad - \lambda_+^{-1} \Lambda^{\frac{1}{2}} \lambda^2 T^- T^2 \otimes \tilde{\partial}^3 \\
&\quad - q \lambda_+^{-1} \Lambda^{\frac{1}{2}} (\tau^1 - \sigma^2) \otimes \tilde{\partial}^3,
\end{aligned} \tag{88}$$

$$S(\tilde{\partial}^3) = -\Lambda^{-\frac{1}{2}}\sigma^2\tilde{\partial}^3 - q^{-\frac{3}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}S^1\partial^+, \quad (89)$$

$$S(\partial^+) = -\Lambda^{-\frac{1}{2}}\tau^1(\tau^3)^{\frac{1}{2}}\partial^+ - q^{\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}T^2(\tau^3)^{\frac{1}{2}}\tilde{\partial}^3,$$

$$S(\partial^-) = -\Lambda^{-\frac{1}{2}}\sigma^2(\tau^3)^{-\frac{1}{2}}\partial^- - q^{-\frac{1}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S^1\partial^0$$

$$+q^{-2}\lambda^2\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S^1T^-\partial^+$$

$$+q^{-\frac{5}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2T^-\tilde{\partial}^3$$

$$-q^{+\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S\tilde{\partial}^3,$$

$$S(\partial^0) = -\Lambda^{-\frac{1}{2}}\tau^1\partial^0 - q^{\frac{5}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}T^2\partial^-$$

$$+q^{-\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}\tau^1T^-\partial^+ + q^{-\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}S^1\partial^+$$

$$+q\lambda_+^{-1}\Lambda^{-\frac{1}{2}}(\sigma^2 - \tau^1)\tilde{\partial}^3 + \lambda_+^{-1}\lambda^2\Lambda^{-\frac{1}{2}}T^2T^-\tilde{\partial}^3,$$

$$\varepsilon(\tilde{\partial}^3) = \varepsilon(\partial^+) = \varepsilon(\partial^-) = \varepsilon(\partial^0) = 0. \quad (90)$$

Similar expressions can be found for the second set of derivatives:

$$\Delta(\hat{\partial}^3) = \hat{\partial}^3 \otimes 1 + \Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2 \otimes \hat{\partial}^3 \quad (91)$$

$$-q^{\frac{3}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}T^2 \otimes \hat{\partial}^-,$$

$$\Delta(\hat{\partial}^-) = \hat{\partial}^- \otimes 1 + \Lambda^{-\frac{1}{2}}\tau^1 \otimes \hat{\partial}^-$$

$$-q^{\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S^1 \otimes \hat{\partial}^3,$$

$$\Delta(\hat{\partial}^+) = \hat{\partial}^+ \otimes 1 + \Lambda^{-\frac{1}{2}}\sigma^2 \otimes \hat{\partial}^+$$

$$-q^{\frac{1}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{-\frac{1}{2}}T^2(\tau^3)^{\frac{1}{2}} \otimes \hat{\partial}^0$$

$$-q^{\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}T^+\sigma^2 \otimes \hat{\partial}^3$$

$$-q^{\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\tau^3T^2 \otimes \hat{\partial}^3$$

$$+q^2\lambda^2\Lambda^{-\frac{1}{2}}T^2T^+ \otimes \hat{\partial}^-,$$

$$\Delta(\hat{\partial}^0) = \hat{\partial}^0 \otimes 1 + \Lambda^{-\frac{1}{2}}(\tau^3)^{\frac{1}{2}}\tau^1 \otimes \hat{\partial}^0$$

$$-q^{-\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}S^1 \otimes \hat{\partial}^+$$

$$-q^{\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{-\frac{1}{2}}(qT^+\tau^1 - T^2) \otimes \hat{\partial}^-$$

$$+\lambda_+^{-1}\lambda^2\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}T^+S^1 \otimes \hat{\partial}^3$$

$$+q^{-1}\lambda_+^{-1}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\tau^3\tau^1 \otimes \hat{\partial}^3$$

$$-q^{-1}\lambda_+^{-1}\Lambda^{-\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2 \otimes \hat{\partial}^3,$$

$$S(\hat{\partial}^3) = -\Lambda^{\frac{1}{2}}\tau^1(\tau^3)^{\frac{1}{2}}\hat{\partial}^3 \quad (92)$$

$$-q^{\frac{3}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{\frac{1}{2}}T^2(\tau^3)^{\frac{1}{2}}\hat{\partial}^-,$$

$$S(\hat{\partial}^-) = -\Lambda^{\frac{1}{2}}\sigma^2\hat{\partial}^- - q^{-\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}S^1\hat{\partial}^3,$$

$$S(\hat{\partial}^+) = -\Lambda^{\frac{1}{2}}\tau^1\hat{\partial}^+ - q^{\frac{5}{2}}\lambda_+^{\frac{1}{2}}\Lambda^{\frac{1}{2}}T^2\hat{\partial}^0$$

$$-q^{\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}(q\tau^1T^+ + T^2)\hat{\partial}^3$$

$$-q^4\lambda^2\Lambda^{\frac{1}{2}}T^2T^+\hat{\partial}^-,$$

$$S(\hat{\partial}^0) = -\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2\hat{\partial}^0$$

$$-q^{-\frac{1}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}S^1\hat{\partial}^+$$

$$-q^{\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\sigma^2T^+\hat{\partial}^-$$

$$+q^{\frac{5}{2}}\lambda_+^{-\frac{1}{2}}\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}\tau^3T^2\hat{\partial}^-$$

$$-\lambda_+^{-1}\lambda^2\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}T^+S^1\hat{\partial}^3$$

$$-q\lambda_+^{-1}\Lambda^{\frac{1}{2}}(\tau^3)^{-\frac{1}{2}}(\sigma^2 - \tau^3\tau^1)\hat{\partial}^3,$$

$$\varepsilon(\hat{\partial}^3) = \varepsilon(\hat{\partial}^+) = \varepsilon(\hat{\partial}^-) = \varepsilon(\hat{\partial}^0) = 0. \quad (93)$$

The Leibniz rules can be read off from the formulae for the co-product as usual, if one knows the representations of the given Lorentz generators and the scaling operator Λ . These representations, however, can be obtained from the commutation relations [21]

$$T^+X^0 = X^0T^+, \quad (94)$$

$$T^+\tilde{X}^3 = \tilde{X}^3T^+ + q^{-\frac{3}{2}}\lambda_+^{\frac{1}{2}}X^+,$$

$$T^+X^+ = q^{-2}X^+T^+,$$

$$T^+X^- = q^2X^-T^+ + q^{-\frac{1}{2}}\lambda_+^{\frac{1}{2}}X^3,$$

$$T^-X^0 = X^0T^-, \quad (95)$$

$$T^-\tilde{X}^3 = \tilde{X}^3T^- + q^{\frac{3}{2}}\lambda_+^{\frac{1}{2}}X^-,$$

$$T^-X^- = q^2X^-T^-,$$

$$T^-X^+ = q^{-2}X^+T^- + q^{\frac{1}{2}}\lambda_+^{\frac{1}{2}}X^3,$$

$$\tau^3X^0 = X^0\tau^3, \quad (96)$$

$$\tau^3\tilde{X}^3 = \tilde{X}^3\tau^3,$$

$$\tau^3X^+ = q^{-4}X^+\tau^3,$$

$$\tau^3X^- = q^4X^-\tau^3,$$

$$T^2\tilde{X}^3 = q^{-1}\tilde{X}^3T^2, \quad (97)$$

$$T^2X^+ = qX^+T^2,$$

$$T^2X^- = q^{-1}X^-T^2 + q^{-\frac{3}{2}}\lambda_+^{-\frac{1}{2}}\tilde{X}^3\tau^1,$$

$$T^2X^3 = qX^3T^2 - q\lambda_+^{-1}\lambda\tilde{X}^3T^2 + q^{-\frac{1}{2}}\lambda_+^{\frac{1}{2}}X^+\tau^1,$$

$$S^1\tilde{X}^3 = q\tilde{X}^3S^1, \quad (98)$$

$$S^1X^- = qX^-S^1,$$

$$\begin{aligned}
 S^1 X^+ &= q^{-1} X^+ S^1 - q^{-\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \tilde{X}^3 \sigma^2, \\
 S^1 X^3 &= q^{-1} X^3 S^1 + q^{-1} \lambda_+^{-1} \lambda \tilde{X}^3 S^1 - q^{\frac{1}{2}} \lambda_+^{-\frac{1}{2}} X^- \sigma^2, \\
 \tau^1 \tilde{X}^3 &= q \tilde{X}^3 \tau^1, \\
 \tau^1 X^- &= q^{-1} X^- \tau^1, \\
 \tau^1 X^+ &= q X^+ \tau^1 - q^{\frac{3}{2}} \lambda_+^{-\frac{1}{2}} \lambda^2 \tilde{X}^3 T^2, \\
 \tau^1 X^3 &= q^{-1} X^3 \tau^1 + q^{-1} \lambda_+^{-1} \lambda \tilde{X}^3 \tau^1 - q^{\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda^2 X^- T^2, \\
 \sigma^2 \tilde{X}^3 &= q^{-1} \tilde{X}^3 \sigma^2, \\
 \sigma^2 X^+ &= q^{-1} X^+ \sigma^2, \\
 \sigma^2 X^- &= q X^- \sigma^2 + q^{\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda^2 \tilde{X}^3 S^1, \\
 \sigma^2 X^3 &= q X^3 \sigma^2 - q \lambda_+^{-1} \lambda \tilde{X}^3 \sigma^2 + q^{-\frac{1}{2}} \lambda_+^{-\frac{1}{2}} \lambda^2 X^+ S^1, \\
 \Lambda X^A &= q^{-2} X^A \Lambda, \quad A = 0, 3, \pm.
 \end{aligned} \tag{100}$$

To calculate representations for partial derivatives and Lorentz generators we need to take special considerations into account. We want to demonstrate this by a short example. By multiple use of the relations (84) and (85), one can show that the following identity holds:

$$\begin{aligned}
 \tilde{\partial}^3 (X^3)^n &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (q^{-2} \lambda^2 \lambda_+^{-1})^k \cdot \sum_{j_1=0}^{n-2k} \sum_{j_2=0}^{n-2k-j_1} \dots \sum_{j_{2k}=0}^{n-2k-j_1-\dots-j_{2k-1}} \\
 &\quad \cdot q^{-2(n-2k)+2(j_2+j_4+\dots+j_{2k})} \\
 &\quad \cdot (X^3 + \lambda \lambda_+^{-1} \tilde{X}^3)^{n-2k} (X^- X^+)^k \tilde{\partial}^3 \\
 &+ q^{-2} \lambda \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (q^{-2} \lambda^2 \lambda_+^{-1})^k \cdot \sum_{j_1=0}^{n-2k-1} \dots \sum_{j_{2k+1}=0}^{n-2k-1-j_1-\dots-j_{2k}} \\
 &\quad \cdot q^{-2(j_1+j_3+\dots+j_{2k+1})} \\
 &\quad \cdot (X^3 + \lambda \lambda_+^{-1} \tilde{X}^3)^{n-2k-1} (X^- X^+)^k X^- \partial^+ \\
 &+ \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (q^{-2} \lambda^2 \lambda_+^{-1})^k \cdot \sum_{j_1=0}^{n-2k-1} \dots \sum_{j_{2k+1}=0}^{n-2k-1-j_1-\dots-j_{2k}} \\
 &\quad \cdot q^{-2(j_1+j_3+\dots+j_{2k+1})} \\
 &\quad \cdot (X^3 + \lambda \lambda_+^{-1} \tilde{X}^3)^{j_1+j_2+\dots+j_{2k+1}} \\
 &\quad \cdot (X^- X^+)^k (X^3)^{n-2k-1-j_1-\dots-j_{2k+1}},
 \end{aligned} \tag{102}$$

where $\lfloor s \rfloor$ denotes the biggest integer not being bigger than s . To go further, one has to overcome two difficulties. The first one is to give normal ordered expressions for $(X^+ X^-)^k$ and $(X^- X^+)^k$. Towards this aim we start from the relations

$$\hat{r}^2 = -a_q(X^0, \tilde{X}^3) + \lambda_+ X^- X^+, \tag{103}$$

$$\hat{r}^2 = -a_{q^{-1}}(X^0, \tilde{X}^3) + \lambda_+ X^+ X^-,$$

where

$$a_q(X^0, \tilde{X}^3) = q^2 (\tilde{X}^3)^2 + q \lambda_+ X^0 \tilde{X}^3, \tag{104}$$

and solve for $X^- X^+$ and $X^+ X^-$. Thus we can write

$$(X^- X^+)^k = (\lambda_+)^{-k} (\hat{r}^2 + a_q(X^0, \tilde{X}^3))^k \tag{105}$$

$$\begin{aligned}
 &= (\lambda_+)^{-k} \sum_{i=0}^k \binom{k}{i} \hat{r}^{2i} (a_q(X^0, \tilde{X}^3))^{k-i} \\
 &= (\lambda_+)^{-k} \sum_{i=0}^k \binom{k}{i} \sum_{p=0}^i \lambda_+^p (X^+)^p \\
 &\quad \cdot (a_q(X^0, q^{2p} \tilde{X}^3))^{k-i} \\
 &\quad \cdot (S_q)_{i,p}(X^0, \tilde{X}^3) (X^-)^p,
 \end{aligned}$$

$$(X^+ X^-)^k = (\lambda_+)^{-k} (\hat{r}^2 + a_{q^{-1}}(X^0, \tilde{X}^3))^k \tag{106}$$

$$\begin{aligned}
 &\cdot (\lambda_+)^{-k} \sum_{i=0}^k \binom{k}{i} \sum_{p=0}^i \lambda_+^p (X^+)^p \\
 &\quad \cdot (a_{q^{-1}}(X^0, q^{2p} \tilde{X}^3))^{k-i} \\
 &\quad \cdot (S_q)_{i,p}(X^0, \tilde{X}^3) (X^-)^p,
 \end{aligned}$$

where

$$\begin{aligned}
 (S_q)_{k,v}(x^0, \tilde{x}^3) &= \begin{cases} 1, & \text{if } v = k, \\ \sum_{j_1=0}^v \sum_{j_2=0}^{j_1} \dots \sum_{j_{k-v}=0}^{j_{k-v-1}} \prod_{l=1}^{k-p} a_q(q^{2j_l} \tilde{x}^3). \end{cases}
 \end{aligned} \tag{107}$$

For the second equality in (105) we have used the fact that \hat{r}^2 and a_q commute. Furthermore, for the third equality in (105) we have inserted the normal ordered expression for powers of \hat{r}^2 which has been taken from [6].

The second problem we have to address has to do with the question: how can we generalize our representations to arbitrary functions? As opposed to the Euclidean cases we cannot rewrite the recursive sums in formula (102) in terms of q -numbers only. However, it should be rather obvious that these recursive sums can be identified with the following quantities:

$$\begin{aligned}
 (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} &= (K_n)_{a_1}^{(k_1)} \circ (K_{n-k_1})_{a_2}^{(k_2)} \circ \dots \circ (K_{n-k_1-\dots-k_l})_{a_l}^{(k_l)},
 \end{aligned} \tag{108}$$

where

$$(K_n)_a^{(k)} \equiv \sum_{j_1=0}^{n-k} \sum_{j_2=0}^{n-k-j_1} \dots \sum_{j_k=0}^{n-k-j_1-\dots-j_{k-1}} a^{j_1+j_2+\dots+j_k} \tag{109}$$

and

$$(K_n)_a^{(k)} \circ (K_{n-k})_b^{(l)} \tag{110}$$

$$\equiv \sum_{j_1=0}^{n-k-l} \dots \sum_{j_{k+l}=0}^{n-k-l-j_1-\dots-j_{k+l-1}} a^{j_1+\dots+j_k} \cdot b^{j_{k+1}+\dots+j_{k+l}}.$$

Now, the point is that these quantities can be used for defining new linear operators, if we require for arbitrary powers of x the following to hold:

$$D_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} x^n = (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} x^{n-k_1-\dots-k_l}.$$

Thus, it remains to derive general formulae for the action of these operators, a problem which is covered in Appendix B.

In principle, we have collected everything needed for writing down representations of the partial derivatives and Lorentz generators. Before doing this let us fix some notation that will be used in the following. First of all we abbreviate

$$(D_{1,q}^3)^{k,l} \equiv D_{1,q^2}^{(k,l)}, \quad (111)$$

$$(D_{2,q}^3)^{k,l} \equiv D_{y_-/x^3, q^2 y_-/x^3}^{(k,l)},$$

$$(D_{3,q}^3)^{k,l} \equiv D_{y_+/x^3, q^2 y_+/x^3, y_-/x^3, q^2 y_-/x^3}^{(k,l,i,j)},$$

where

$$y_{\pm} = y_{\pm}(x^0, \tilde{x}^3) = x^0 + \frac{2q^{\pm 1}}{\lambda_+} \tilde{x}^3. \quad (112)$$

It should be clear that these operators have to act on the coordinate x^3 only. Additionally, we make the following definitions:

$$(M^{\pm})_{i,j}^k(\underline{x}) \equiv (M^{\pm})_{i,j}^k(x^0, x^+, \tilde{x}^3, x^-) \quad (113)$$

$$= \binom{k}{i} \lambda_+^j (a_{q^{\pm 1}}(q^{2j} \tilde{x}^3))^i (x^+ x^-)^j (S_q)_{k-j,j}(x^0, \tilde{x}^3),$$

$$(M^{+-})_{i,j,u}^{k,l}(\underline{x}) \equiv (M^{+-})_{i,j,u}^{k,l}(x^0, x^+, \tilde{x}^3, x^-) \quad (114)$$

$$= \binom{k}{i} \binom{l}{j} \lambda_+^u (a_q(q^{2u} \tilde{x}^3))^{k-i} (a_{q^{-1}}(q^{2u} \tilde{x}^3))^{l-j} \cdot (x^+ x^-)^u (S_q)_{i+j,u}(x^0, \tilde{x}^3).$$

Notice that in what follows the normal ordering for which our formulae shall work is indicated by the sequence of coordinates the given functions depend on. In this way the representations of the conjugated partial derivatives become

$$\hat{\partial}^3 \triangleright f(x^+, \tilde{x}^3, x^3, x^-) \quad (115)$$

$$= \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \leq i+j \leq k} (M^-)_{i,j}^k(\underline{x}) (\tilde{T}^3)_j^i f,$$

$$\hat{\partial}^- \triangleright f(x^+, \tilde{x}^3, x^3, x^-) \quad (116)$$

$$= -q^{-1} D_{q^2}^+ f + \frac{\lambda}{\lambda_+} \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \leq i+j \leq k} \left\{ (M^+)_{i,j}^k(\underline{x}) (T_1^-)_j^i f \right.$$

$$\left. + q^{-1} (M^-)_{i,j}^k(\underline{x}) (T_2^-)_j^i f \right\},$$

$$\hat{\partial}^+ \triangleright f(x^+, \tilde{x}^3, x^3, x^-) \quad (117)$$

$$= -q \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \leq i+j \leq k} \left\{ (M^-)_{i,j}^k(\underline{x}) (T_1^+)_j^i f \right.$$

$$\left. + \lambda (M^+)_{i,j}^k(\underline{x}) (T_2^+)_j^i f \right\}$$

$$- q \frac{\lambda}{\lambda_+} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l}$$

$$\cdot \sum_{i=0}^k \sum_{j=0}^l \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k,l}(\underline{x}) (T_3^+)_u^{k,l} f$$

$$- \frac{\lambda}{\lambda_+} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l+1}$$

$$\cdot \sum_{i=0}^k \sum_{j=0}^{l+1} \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k,l+1}(\underline{x}) (T_4^+)_u^{k,l} f,$$

$$\hat{\partial}^0 \triangleright f(x^+, \tilde{x}^3, x^3, x^-) \quad (118)$$

$$= \sum_{k=0}^{\infty} \alpha_+^k \sum_{0 \leq i+j \leq k} \left\{ (M^+)_{i,j}^k(\underline{x}) (T_1^0)_j^i f \right.$$

$$\left. - q \frac{\lambda}{\lambda_+} (M^-)_{i,j}^k(\underline{x}) (T_2^0)_j^i f \right\}$$

$$- q^2 \frac{\lambda}{\lambda_+} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l}$$

$$\cdot \sum_{i=0}^k \sum_{j=0}^l \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k,l}(\underline{x}) (T_3^0)_u^{k,l} f$$

$$+ \frac{\beta}{\lambda_+} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l+1}$$

$$\cdot \sum_{i=0}^{k+1} \sum_{j=0}^l \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k+1,l}(\underline{x}) (T_4^0)_u^{k,l} f$$

$$+ \lambda_+^{-1} \sum_{0 \leq k+l < \infty} \alpha_+^{k+l+1}$$

$$\cdot \sum_{i=0}^k \sum_{j=0}^{l+1} \sum_{0 \leq u \leq i+j} (M^{+-})_{i,j,u}^{k,l+1}(\underline{x}) (T_5^0)_u^{k,l} f,$$

where

$$\alpha_+ = -q^2 \frac{\lambda^2}{\lambda_+^2}, \quad \beta = q + \lambda_+. \quad (119)$$

To get expressions with a more obvious structure we have introduced the abbreviations

$$(T^{3/0})_j^i f = \left[(O^{3/0})_i f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2j} x^{3/0}), \quad (120)$$

$$(T_1^-)_j^i f = \left[(O_1^-)_i f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2(j+1)} x^{3/0}), \quad (121)$$

$$(T_2^-)_j^i f = \left[(O_2^-)_i f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2j} x^{3/0}),$$

$$(T_1^0)_j^i f \quad (122)$$

$$= \left[(O_1^0)_i f \Big|_{x^3 \rightarrow y_+} - q^2 \frac{\lambda}{\lambda_+} (O_2^0)_i f \Big|_{x^3 \rightarrow q^2 y_+} \right] (q^{2j} x^{3/0}),$$

$$(T_2^0)_j^i f = \left[(O_3^0)_i f \Big|_{x^3 \rightarrow x^0 + x^3/0} + (O_4^0)_i f \Big|_{x^3 \rightarrow q^2 y_-} \right] (q^{2j} x^{3/0}),$$

$$(T_3^0)_u^{k,l} f = \left[(Q_1^0)_{k,l} f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2u} x^{3/0}),$$

$$(T_4^0)_u^{k,l} f = \left[(Q_2^0)_{k,l} f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2u} x^{3/0}),$$

$$(T_5^0)_u^{k,l} f = \left[(Q_3^0)_{k,l} f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2u} x^{3/0}),$$

$$(T_1^+)_j^i f = \left[(O_1^+)_i f \Big|_{x^3 \rightarrow q^2 y_-} \right] (q^{2j} x^{3/0}), \quad (123)$$

$$(T_2^+)_j^i f = \left[(O_2^+)_i f \Big|_{x^3 \rightarrow y_+} \right] (q^{2j} x^{3/0}),$$

$$(T_3^+)_u^{k,l} f = \left[(Q_1^+)_{k,l} f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2u} x^{3/0}),$$

$$(T_4^+)_u^{k,l} f = \left[(Q_2^+)_{k,l} f \Big|_{x^3 \rightarrow x^0 + x^3/0} \right] (q^{2u} x^{3/0}),$$

which, in turn, depend on the following operators:

$$(\tilde{O}^3)_{k,l} f = (D_{2,q}^3)^{k,k+1} f(q^2 x^+), \quad (124)$$

$$(O_1^-)_{k,l} f = x^- (D_{2,q}^3)^{k+1,k+1} f(q^2 x^+), \quad (125)$$

$$(O_2^-)_{k,l} f = \tilde{x}^3 D_{q^2}^+ (D_{2,q}^3)^{k,k+1} f(q^2 x^+),$$

$$(O_1^0)_{k,l} f = \tilde{D}_{q^2}^3 (D_{1,q}^3)^{k,k} f \quad (126)$$

$$-q^3 \lambda_+^{-1} \lambda^2 x^+ \tilde{x}^3 D_{q^2}^+ \tilde{D}_{q^2}^3 (D_{1,q}^3)^{k,k+1} f,$$

$$(O_2^0)_{k,l} f = x^- (D_{1,q^{-1}}^3)^{k,k+1} D_{q^2}^- f(q^2 \tilde{x}^3),$$

$$(O_3^0)_{k,l} f = q x^+ D_{q^2}^+ (D_{2,q}^3)^{k,k+1} f,$$

$$(O_4^0)_{k,l} f = \tilde{x}^3 D_{q^2}^+ (D_{1,q^{-1}}^3)^{k,k} D_{q^2}^- f,$$

$$(Q_1^0)_{k,l} f = (x^0 + q^{-1} \lambda x^3) (D_{3,q}^3)_{l,l+1}^{k+1,k} f \quad (127)$$

$$+ q \lambda_+^{-1} \lambda (q + \lambda_+) x^+ \tilde{x}^3 D_{q^2}^+ (D_{3,q}^3)_{l,l+1}^{k,k+1} f$$

$$- q^3 \lambda_+^{-1} \lambda^2 x^+ \tilde{x}^3 (x^0 + q^{-1} \lambda \tilde{x}^3)$$

$$\cdot D_{q^2}^+ (D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_2^0)_{k,l} f = (D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_3^0)_{k,l} f = q^{-1} (D_{3,q}^3)_{l+1,l+1}^{k+1,k} f$$

$$- q^2 \lambda_+^{-1} \lambda^2 x^+ \tilde{x}^3 D_{q^2}^+ (D_{3,q}^3)_{l+1,l+1}^{k+1,k+1} f,$$

$$(O_1^+)_{k,l} f = (D_{1,q^{-1}}^3)^{k,k} D_{q^2}^- f, \quad (128)$$

$$(O_2^+)_{k,l} f = x^+ \tilde{D}_{q^2}^3 (D_{1,q}^3)^{k,k+1} f,$$

$$(Q_1^+)_{k,l} f = (q + \lambda_+) x^+ (D_{3,q}^3)_{l,l+1}^{k,k+1} f \quad (129)$$

$$- q^2 \lambda x^+ (x^0 + q^{-1} \lambda x^3) (D_{3,q}^3)_{l,l+1}^{k+1,k+1} f,$$

$$(Q_2^+)_{k,l} f = x^+ (D_{3,q}^3)_{l+1,l+1}^{k+1,k+1} f.$$

In the same way the representations of the Lorentz generators are explicitly given by

$$\Lambda \triangleright f(x^+, x^0, \tilde{x}^3, x^-) \quad (130)$$

$$= f(q^{-2} x^+, q^{-2} x^0, q^{-2} \tilde{x}^3, q^{-2} x^-),$$

$$\tau^3 \triangleright f(x^+, x^0, \tilde{x}^3, x^-) \quad (131)$$

$$= f(q^{-4} x^+, q^4 x^-),$$

$$T^+ \triangleright f(x^+, x^0, \tilde{x}^3, x^-) \quad (132)$$

$$= q^{-\frac{1}{2}} \lambda_+^{\frac{1}{2}} \left[x^0 D_{q^2}^- f + \tilde{x}^3 D_{q^4}^- f + q x^+ \tilde{D}_{q^2}^3 f \right] (q^{-2} x^+),$$

$$T^- \triangleright f(x^+, x^0, \tilde{x}^3, x^-) \quad (133)$$

$$= q^{\frac{1}{2}} \lambda_+^{\frac{1}{2}} \left[x^0 D_{q^2}^+ f + \tilde{x}^3 D_{q^4}^+ f + q x^- \tilde{D}_{q^2}^3 f \right] (q^{-2} x^+),$$

$$T^2 \triangleright f(x^+, x^3, \tilde{x}^3, x^-) \quad (134)$$

$$= \lambda_+^{-\frac{1}{2}} \sum_{k=0}^{\infty} \alpha_0^k \sum_{0 \leq i+j \leq k} \left\{ q^{\frac{1}{2}} (M^-)_{i,j}^k(\underline{x}) (T_1^T)_{j,j}^i f \right. \\ \left. + q^{-\frac{1}{2}} (M^+)_{i,j}^k(\underline{x}) (T_2^T)_{j,j}^i f \right\},$$

$$S^1 \triangleright f(x^+, x^3, \tilde{x}^3, x^-) \quad (135)$$

$$= -q \lambda_+^{-\frac{1}{2}} \sum_{k=0}^{\infty} \alpha_0^k \sum_{0 \leq i+j \leq k} \left\{ q^{\frac{1}{2}} (M^-)_{i,j}^k(\underline{x}) (T_1^S)_{j,j}^i f \right. \\ \left. + q^{-\frac{1}{2}} (M^+)_{i,j}^k(\underline{x}) (T_2^S)_{j,j}^i f \right\},$$

$$\tau^1 \triangleright f(x^+, x^3, \tilde{x}^3, x^-) \quad (136)$$

$$= \sum_{k=0}^{\infty} \alpha_0^k \sum_{0 \leq i+j \leq k} \left\{ (M^+)_{i,j}^k(\underline{x}) (T_2^T)_{j,j}^i f \right. \\ \left. - \frac{\lambda^2}{\lambda_+} (M^-)_{i,j}^k(\underline{x}) (T_1^T)_{j,j}^i f \right\},$$

$$\sigma^2 \triangleright f(x^+, x^3, \tilde{x}^3, x^-) \quad (137)$$

$$= \sum_{k=0}^{\infty} \alpha_0^k \sum_{0 \leq i+j \leq k} (M^-)_{i,j}^k(\underline{x})(T^\sigma)_j^i f,$$

where $\alpha_0 = -(\lambda/\lambda_+)^2$. For the purpose of abbreviation we have again set

$$(T_1^T)_j^k \triangleright f = \left[(O_1^T)_k f \Big|_{x^3 \rightarrow y_-} \right] (q^{2j} x^{3/0}), \quad (138)$$

$$(T_2^T)_j^k \triangleright f = \left[(O_2^T)_k f \Big|_{x^3 \rightarrow y_+} \right] (q^{2j} x^{3/0}), \quad (139)$$

$$(T_1^S)_j^k \triangleright f = \left[(O_1^S)_k f \Big|_{x^3 \rightarrow y_-} \right] (q^{2j} x^{3/0}), \quad (140)$$

$$(T_2^S)_j^k \triangleright f = \left[(O_2^S)_k f \Big|_{x^3 \rightarrow y_+} \right] (q^{2j} x^{3/0}), \quad (141)$$

$$(T_1^\tau)_j^k \triangleright f = \left[(O_1^\tau)_k f \Big|_{x^3 \rightarrow y_-} \right] (q^{2j} x^{3/0}), \quad (142)$$

$$(T_2^\tau)_j^k \triangleright f = \left[(O^\sigma)_k f \Big|_{x^3 \rightarrow y_-} \right] (q^{2j} x^{3/0}), \quad (143)$$

where

$$(O_1^T)_k f = q^{-1} \tilde{x}^3 (D_{1,q^{-1}}^3)^{k,k} \cdot D_{q^2}^- f(qx^+, q^{-1} \tilde{x}^3, q^{-1} x^-), \quad (144)$$

$$(O_2^T)_k f = x^+ (D_{1,q}^3)^{k,k+1} f(qx^+, q\tilde{x}^3, q^{-1} x^-),$$

$$(O_1^S)_k f = q^{-1} \tilde{x}^3 (D_{1,q^{-1}}^3)^{k,k} \cdot D_{q^2}^+ f(q^{-1} x^+, q^{-1} \tilde{x}^3, qx^-), \quad (145)$$

$$(O_2^S)_k f = x^- (D_{1,q^{-1}}^3)^{k,k+1} f(q^{-1} x^+, q\tilde{x}^3, qx^-),$$

$$(O_1^\tau)_k f = (\tilde{x}^3)^2 D_{q^2}^+ (D_{1,q^{-1}}^3)^{k,k} \cdot D_{q^2}^- f(qx^+, q^{-1} \tilde{x}^3, q^{-1} x^-), \quad (146)$$

$$(O_2^\tau)_k f = (D_{1,q}^3)^{k,k} f(qx^+, q\tilde{x}^3, q^{-1} x^-) - q\lambda_+^{-1} \lambda^2 x^+ \tilde{x}^3 D_{q^2}^+ \cdot (D_{1,q}^3)^{k,k+1} f(qx^+, q\tilde{x}^3, q^{-1} x^-),$$

$$(O_3^\tau)_k f = q\tilde{x}^3 x^- (D_{1,q^{-1}}^3)^{k,k+1} \cdot D_{q^2}^- f(qx^+, q\tilde{x}^3, q^{-1} x^-),$$

$$(O^\sigma)_k f = (D_{1,q^{-1}}^3)^{k,k} f(q^{-1} x^+, q^{-1} \tilde{x}^3, qx^-). \quad (147)$$

Now we deal with representations for the partial derivatives ∂^μ , $\mu = 0, \pm, \tilde{3}$ which can directly be obtained from representations for the conjugated ones, if we apply the following transformation:

$$\partial^\pm \tilde{\triangleright} f \stackrel{\pm \leftrightarrow \mp}{\longleftarrow \frac{1}{q}} \hat{\partial}^\mp \triangleright f, \quad (148)$$

$$\partial^{\tilde{3}} \tilde{\triangleright} f \stackrel{\pm \leftrightarrow \mp}{\longleftarrow \frac{1}{q}} \hat{\partial}^{\tilde{3}} \triangleright f,$$

which means concretely that the substitutions

$$D_{q^2}^\pm \rightarrow D_{q^2}^\mp, \quad \hat{n}_\pm \rightarrow -\hat{n}_\mp, \quad q^{\pm 1} \rightarrow q^{\mp 1} \quad (149)$$

interchange the different representations. It is very important to notice that the representations on the left hand side of (148) have to refer to a different normal ordering given by $X^- X^3 \tilde{X}^3 X^+$. However, by the identities

$$\partial^\mu \triangleright (\hat{U}^{-1} f) = \hat{U}^{-1} (\partial^\mu \tilde{\triangleright} f), \quad (150)$$

$$\hat{\partial}^\mu \tilde{\triangleright} (\hat{U} f) = \hat{U} (\hat{\partial}^\mu \triangleright f),$$

with

$$\hat{U}^{-1} f \quad (151)$$

$$= \sum_{i=0}^{\infty} \left(\frac{\lambda}{\lambda_+} \right)^i \sum_{k+j=i} \frac{(R_q)_{k,j}(\underline{x})}{[[k]]_{q^2}! [[j]]_{q^2}!} \cdot q^{2\hat{n}_+ \hat{n}_- + (\hat{n}_+ + \hat{n}_-)(2\hat{n}_3 + i) + 2\hat{n}_3 i} \cdot \left((D_{q^2}^+)^i (D_{q^2}^-)^i f \right) (x^0, q^{j-k} x^+, \tilde{x}^3, q^{j-k} x^-),$$

$$\hat{U} f \quad (152)$$

$$= \sum_{i=0}^{\infty} \left(-\frac{\lambda}{\lambda_+} \right)^i \sum_{k+j=i} \frac{(R_{q^{-1}})_{k,j}(\underline{x})}{[[k]]_{q^{-2}}! [[j]]_{q^{-2}}!} \cdot q^{-2\hat{n}_+ \hat{n}_- - (\hat{n}_+ + \hat{n}_-)(2\hat{n}_3 + i) - 2\hat{n}_3 i} \cdot \left((D_{q^{-2}}^+)^i (D_{q^{-2}}^-)^i f \right) (x^0, q^{k-j} x^+, \tilde{x}^3, q^{k-j} x^-)$$

the representations for ordering $X^- X^3 \tilde{X}^3 X^+$ can be transformed into those for reversed ordering $X^+ X^3 \tilde{X}^3 X^-$. The reason for the existence of (148) should be clear from the fact that the substitutions

$$\tilde{\partial}^{\tilde{3}} \rightarrow \hat{\partial}^{\tilde{3}}, \quad (153)$$

$$\partial^0 \rightarrow \hat{\partial}^0,$$

$$\partial^\pm \rightarrow \hat{\partial}^\mp,$$

$$X^\pm \rightarrow X^\mp,$$

$$q \rightarrow q^{-1}$$

interchange the relations (79)–(82) and (84)–(87).

Finally, from the conjugation properties [10]

$$\overline{X^0} = X^0, \quad (154)$$

$$\overline{\tilde{X}^3} = \tilde{X}^3,$$

$$\overline{X^\pm} = -q^{\pm 1} X^\mp,$$

$$\overline{\partial^{\tilde{3}}} = -q^4 \hat{\partial}^{\tilde{3}}, \quad (155)$$

$$\overline{\partial^0} = -q^4 \hat{\partial}^0,$$

$$\begin{aligned} \overline{\partial^\mp} &= -q^5 \hat{\partial}^-, \\ \overline{\partial^\mp} &= -q^3 \hat{\partial}^+, \\ \overline{T^\mp} &= q^{\mp 2} T^\mp \end{aligned} \tag{156}$$

we can derive the following rules for transforming right and left representations into each other:

$$f \triangleleft \partial^0 \xleftrightarrow{\pm\leftrightarrow\mp} -q^4 \hat{\partial}^0 \triangleright f, \tag{157}$$

$$f \triangleleft \tilde{\partial}^3 \xleftrightarrow{\pm\leftrightarrow\mp} -q^4 \hat{\tilde{\partial}}^3 \triangleright f,$$

$$f \triangleleft \partial^\mp \xleftrightarrow{\pm\leftrightarrow\mp} -q^4 \hat{\partial}^\mp \triangleright f,$$

$$f \triangleleft \hat{\partial}^0 \xleftrightarrow{\pm\leftrightarrow\mp} -q^{-4} \partial^0 \triangleright f, \tag{158}$$

$$f \triangleleft \hat{\tilde{\partial}}^3 \xleftrightarrow{\pm\leftrightarrow\mp} -q^{-4} \tilde{\partial}^3 \triangleright f,$$

$$f \triangleleft \hat{\partial}^\pm \xleftrightarrow{\pm\leftrightarrow\mp} -q^{-4} \partial^\mp \triangleright f,$$

$$f \triangleleft T^\pm \xleftrightarrow{\pm\leftrightarrow\mp} -q^{\mp 3} T^\mp \triangleright f, \tag{159}$$

where the symbol $\xleftrightarrow{\pm\leftrightarrow\mp}$ has the same meaning as in Sect. 2. Once again the simplest way to determine the right representations of the remaining generators is described by the identity

$$f \triangleleft h = S^{-1}(h) \triangleright f. \tag{160}$$

With the Hopf structure of the Lorentz generators at hand [11] we end up with the expression

$$f \triangleleft T^2 = -(\tau^3)^{\frac{1}{2}} T^2 \triangleright f, \tag{161}$$

$$f \triangleleft S^1 = -q^2 (\tau^3)^{-\frac{1}{2}} S^1 \triangleright f,$$

$$f \triangleleft \tau^1 = \sigma^2 \triangleright f, \tag{162}$$

$$f \triangleleft \sigma^2 = \tau^1 \triangleright f,$$

$$f \triangleleft \tau^3 = (\tau^3)^{-1} \triangleright f, \tag{163}$$

$$f \triangleleft \Lambda = \Lambda^{-1} \triangleright f.$$

5 Remarks

Let us end with a few comments on our representations. First of all, from a physical point of view partial derivatives are objects generating translations in time or space. According to

$$\partial^A f = (\partial^A f)_0 + \sum_{i>0} \lambda^i (\partial^A f)_i, \tag{164}$$

their representations can be divided up into one part reducing to ordinary derivatives in the undeformed limit ($q = 1$) and a second part of correction terms disappearing in that case. The existence of the correction terms can be well understood, if one assumes that non-commutativity results

from a coupling of the different directions in space. Thus a flow of momentum in only one direction is in general not possible and the corrections should be responsible for this feature. So far we can sum up that the situation in non-commutative spaces seems to be similar to that of solids. In fact, if such a solid state has to undergo a deformation in some direction, the other directions will also be influenced due to their coupling.

A Notation

(1) The q -number is defined by [12]

$$[[c]]_{q^a} \equiv \frac{1 - q^{ac}}{1 - q^a}, \quad a, c \in \mathbb{C}. \tag{165}$$

(2) For $m \in \mathbb{N}$, we can introduce the q -factorial by setting

$$[[m]]_{q^a}! \equiv [[1]]_{q^a} [[2]]_{q^a} \dots [[m]]_{q^a}, \tag{166}$$

$$[[0]]_{q^a}! \equiv 1.$$

(3) There is also a q -analogue of the usual binomial coefficients, the so-called q -binomial coefficients defined by the formula

$$\begin{aligned} & \begin{bmatrix} \alpha \\ m \end{bmatrix}_{q^a} \\ & \equiv \frac{[[\alpha]]_{q^a} [[\alpha - 1]]_{q^a} \dots [[\alpha - m + 1]]_{q^a}}{[[m]]_{q^a}!} \end{aligned} \tag{167}$$

where $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$.

(4) Commutative co-ordinates are usually denoted by small letters (e.g. x^+ , x^- , etc.), non-commutative co-ordinates in capital (e.g. X^+ , X^- , etc.).

(5) Note that in functions only such variables are explicitly displayed which are affected by a scaling. For example, we write

$$f(q^2 x^+)$$

instead of

$$f(q^2 x^+, x^3, x^-).$$

(6) Arguments enclosed in parentheses shall refer to the first object on their left. For example, we have

$$D_{q^2}^+ f(q^2 x^+) = D_{q^2}^+ (f(q^2 x^+))$$

or

$$D_{q^2}^+ [D_{q^2}^+ f + D_{q^2}^- f](q^2 x^+) = D_{q^2}^+ ([D_{q^2}^+ f + D_{q^2}^- f](q^2 x^+)).$$

However, the symbol $|_{x' \rightarrow x}$ applies to the whole expression on its left side reaching up to the next opening bracket or \pm sign.

(7) The Jackson derivative referring to the coordinate x^A is defined by

$$D_{q^a}^A f \equiv \frac{f(x^A) - f(q^a x^A)}{(1 - q^a)x^A}, \tag{168}$$

where f may depend on other coordinates as well. Higher Jackson derivatives are obtained by applying the above operator $D_{q^a}^A$ several times:

$$(D_{q^a}^A)^i f \equiv \underbrace{D_{q^a}^A D_{q^a}^A \dots D_{q^a}^A}_{i \text{ times}} f. \tag{169}$$

(8) Additionally, we need operators of the following form:

$$\hat{n}^A \equiv x^A \frac{\partial}{\partial x^A}. \tag{170}$$

B New Jackson derivatives

In the calculations of Sect. 4 we have introduced the following quantities:

$$(K_n)_a^{(k)} \equiv \sum_{j_1=0}^{n-k} \sum_{j_2=0}^{n-k-j_1} \dots \sum_{j_k=0}^{n-k-j_1-\dots-j_{k-1}} a^{j_1+j_2+\dots+j_k}. \tag{171}$$

Additionally, we have defined an operation \circ by

$$\begin{aligned} & (K_n)_a^{(k)} \circ (K_{n-k})_b^{(l)} \tag{172} \\ & \equiv \sum_{j_1=0}^{n-k-l} \dots \sum_{j_{k+l}=0}^{n-k-l-j_1-\dots-j_{k+l-1}} a^{j_1+\dots+j_k} \cdot b^{j_{k+1}+\dots+j_{k+l}}, \end{aligned} \tag{3}$$

leading to new expressions denoted by

$$\begin{aligned} & (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} \tag{173} \\ & \equiv (K_n)_{a_1}^{(k_1)} \circ (K_{n-k_1})_{a_2}^{(k_2)} \circ \dots \circ (K_{n-k_1-\dots-k_l})_{a_l}^{(k_l)}. \end{aligned}$$

Furthermore these new quantities show a number of simple properties, for instance,

$$(1) \quad (K_n)_1^{(k)} = \binom{n}{k}, \tag{174}$$

$$(2) \quad (K_n)_{a,a}^{(k,l)} = (K_n)_a^{(k+l)}, \tag{175}$$

$$(3) \quad (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} = (K_n)_{a_{\pi(1)}, \dots, a_{\pi(l)}}^{(k_{\pi(1)}, \dots, k_{\pi(l)})}, \tag{176}$$

where π is any permutation of the $\{1, \dots, l\}$. Due to these properties the quantities of (173) can be divided into three different sets:

$$(1) \quad (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)}, \tag{177}$$

with $a_i \neq 1$ for all $i \in \{1, \dots, l\}$,

$$(2) \quad (K_n)_{a_1, \dots, a_{l-1}, 1}^{(k_1, \dots, k_l)}, \tag{178}$$

with $a_i \neq 1$ for all $i \in \{1, \dots, l-1\}$,

$$(3) \quad (K_n)_1^{(k)}.$$

It is now our aim to show that each $(K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)}$ can be expressed in terms of the binomials $(K_n)_1^{(k)}$. Towards this end we need the following additional rules:

$$(1) \quad (K_n)_a^{(k)} = (1-a)^{-k} - \sum_{m=0}^{k-1} a^{n-m} (1-a)^{m-k} (K_n)_1^{(m)} \tag{179}$$

$$(2) \quad (K_n)_{a,1}^{(k,l)} \tag{180}$$

$$\begin{aligned} & = \sum_{m=0}^l (-1)^{l-m} \binom{k+l-1-m}{k-1} \\ & \quad \cdot (1-a)^{m-k-l} (K_n)_1^{(m)} \\ & \quad + (-1)^{l+1} \sum_{m=0}^{k-1} a^{n-m} \binom{k+l-1-m}{l} \\ & \quad \cdot (1-a)^{m-k-l} (K_n)_1^{(m)}, \end{aligned}$$

$$\begin{aligned} & (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} \circ \left((b)^{n-k_1-\dots-k_l} (K_{n-k_1-\dots-k_l})_{c_1, \dots, c_p}^{(q_1, \dots, q_p)} \right) \tag{181} \\ & = b^{n-k_1-\dots-k_l} \left((K_n)_{\frac{a_1}{b}, \dots, \frac{a_l}{b}}^{(k_1, \dots, k_l)} \circ (K_{n-k_1-\dots-k_l})_{c_1, \dots, c_p}^{(q_1, \dots, q_p)} \right), \end{aligned}$$

which can be verified in a quite elementary way. From these rules we find a recursion relation; we have

$$\begin{aligned} & (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} \tag{182} \\ & = (K_n)_{a_1, \dots, a_{l-1}}^{(k_1, \dots, k_{l-1})} \circ (K_{n-k_1-\dots-k_{l-1}})_{a_l}^{(k_l)} \\ & = - \sum_{m=0}^{k_l-1} a_l^{n-k_1-\dots-k_{l-1}-m} \cdot (1-a_l)^{m-k_l} (K_n)_{\frac{a_1}{a_l}, \dots, \frac{a_{l-1}}{a_l}, 1}^{(k_1, \dots, k_{l-1}, m)} \\ & \quad + (1-a_l)^{-k_l} (K_n)_{a_1, \dots, a_{l-1}}^{(k_1, \dots, k_{l-1})}. \end{aligned}$$

Using this relation repeatedly the quantities of the first set can now be reduced to those of the second one, as one gets

$$\begin{aligned} & (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} \tag{183} \\ & = - \sum_{i=1}^l \left(\prod_{j=i+1}^l (1-a_j)^{-k_j} \right) \end{aligned}$$

$$\cdot \sum_{m=0}^{k_i-1} a_i^{n-k_1-\dots-k_{i-1}-m} \cdot (1-a_i)^{m-k_i} (K_n)_{\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, 1}^{(k_1, \dots, k_{i-1}, m)}.$$

$D_{a_1, \dots, a_l}^{(k_1, \dots, k_l)}$ can always be expressed in terms of the simple operators

$$D_1^{(k)} f(x) = \frac{1}{k!} \frac{\partial^k}{\partial x^k} f(x). \tag{188}$$

In the same way the quantities of the second set can, in turn, be reduced to those of the last one by applying another recursion relation given by

$$\begin{aligned} & (K_n)_{a_1, \dots, a_{l-1}, 1}^{(k_1, \dots, k_l)} \tag{184} \\ &= (K_n)_{a_1, \dots, a_{l-2}}^{(k_1, \dots, k_{l-2})} \circ (K_{n-k_1-\dots-k_{l-2}})_{a_{l-1}, 1}^{(k_{l-1}, k_l)} \\ &= \sum_{m=0}^{k_l} (-1)^{k_l-m} \binom{k_l + k_{l-1} - m - 1}{k_{l-1} - 1} \\ &\quad \cdot (1 - a_{l-1})^{m-k_l-k_{l-1}} (K_n)_{a_1, \dots, a_{l-2}, 1}^{(k_1, \dots, k_{l-2}, m)} \\ &\quad - (-1)^{k_l} \sum_{m=0}^{k_{l-1}-1} \binom{k_l + k_{l-1} - m - 1}{k_l} \\ &\quad \cdot a_{l-1}^{n-k_1-\dots-k_{l-2}-m} (1 - a_{l-1})^{m-k_l-k_{l-1}} \\ &\quad \cdot (K_n)_{\frac{a_1}{a_{l-1}}, \dots, \frac{a_{l-2}}{a_{l-1}}, 1}^{(k_1, \dots, k_{l-2}, m)}. \end{aligned}$$

If we introduce operators acting on powers x^n by

$$D_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} x^n = \begin{cases} (K_n)_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} x^{n-k_1-\dots-k_l}, \\ 0, & \text{if } n < k_1 + \dots + k_l, \end{cases} \tag{185}$$

the relations (183) and (184) correspond to the identities

$$\begin{aligned} & D_{a_1, \dots, a_l}^{(k_1, \dots, k_l)} f(x) \tag{186} \\ &= - \sum_{i=1}^l \left(\prod_{j=i+1}^l (x - a_j x)^{-k_j} \right) \cdot \sum_{m=0}^{k_i-1} (x - a_i x)^{m-k_i} \\ &\quad \cdot \left(D_{\frac{a_1}{a_i}, \dots, \frac{a_{i-1}}{a_i}, 1}^{(k_1, \dots, k_{i-1}, m)} f \right) (a_i x), \end{aligned}$$

where $a_i \neq 1$ for all $i \in \{1, \dots, l\}$, and

$$\begin{aligned} & D_{a_1, \dots, a_{l-1}, 1}^{(k_1, \dots, k_l)} f(x) \tag{187} \\ &= \sum_{m=0}^{k_l} (-1)^{k_l-m} \binom{k_l + k_{l-1} - m - 1}{k_{l-1} - 1} \\ &\quad \cdot (x - a_{l-1} x)^{m-k_l-k_{l-1}} D_{a_1, \dots, a_{l-2}, 1}^{(k_1, \dots, k_{l-2}, m)} f(x) \\ &\quad - (-1)^{k_l} \sum_{m=0}^{k_{l-1}-1} \binom{k_l + k_{l-1} - m - 1}{k_l} \\ &\quad \cdot (x - a_{l-1} x)^{m-k_l-k_{l-1}} \left(D_{\frac{a_1}{a_{l-1}}, \dots, \frac{a_{l-2}}{a_{l-1}}, 1}^{(k_1, \dots, k_{l-2}, m)} f \right) (a_{l-1} x), \end{aligned}$$

where $a_i \neq 1$ for all $i \in \{1, \dots, l-1\}$. With these formulae at hand one readily checks that the derivative operators

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